

## A HARNACK TYPE INEQUALITY FOR CERTAIN COMPLEX MONGE-AMPERÉ EQUATIONS

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### 0. Introduction

In finding a Kähler-Einstein metric on a compact Kähler manifold  $(M, g)$  with  $C_1(M) > 0$ , one needs to solve the following complex Monge-Ampère equations:

$$(*)_t \quad \begin{aligned} (\omega_g + \partial\bar{\partial}\phi)^n &= e^{f-t\phi} \omega_g^n, \\ \omega_g + \partial\bar{\partial}\phi &> 0, \quad \text{on } M, \end{aligned}$$

where  $\omega_g$  is the Kähler form associated with the metric  $g$ ,  $\omega_g^n = \omega_g \wedge \cdots \wedge \omega_g$  is the volume form,  $0 \leq t \leq 1$ ,  $\partial\bar{\partial}f = \text{Ric}(g) - \omega_g$ ,  $\int_M e^f \omega_g^n = \int_M \omega_g^n = \text{Vol}_g(M)$ , and  $n = \dim M$ .

While the prior estimates of higher derivatives have been obtained by Yau [8] for the solutions of  $(*)_t$  more than ten years ago, little is known about the supreme norms of the solutions of  $(*)_t$ . In [7], the author proved that  $-\inf_M \phi$  is bounded from above by  $n \sup_M \phi + C$  for any solution  $\phi$  of  $(*)_t$ , where  $C$  is a constant independent of  $\phi$ . Actually, it is implied in the proof there that  $C$  depends only on  $t$ . By a completely different method, Siu also proved a slightly weaker version of the above inequality [6], i.e., for any  $\varepsilon > 0$ , there is a constant  $C_\varepsilon$ , depending on the metric  $g$ ,  $t$  and  $\varepsilon$ , such that for any solution  $\phi$  of  $(*)_t$   $-\inf_M \phi \leq (n + \varepsilon) \sup_M \phi + C_\varepsilon$ . In this note, we develop the idea in the proof of the above Harnack type inequality in [7] and prove the following theorem.

**Theorem 1.** *Let  $(M, g)$  be a compact Kähler manifold with  $C_1(M) > 0$ ,  $n = \dim M$ . Then for any  $\psi \in C^2(M, \mathbb{R})$  with  $\omega_g + \partial\bar{\partial}\psi \geq 0$  and  $\int_M e^{f-t\psi} \omega_g^n = \text{Vol}_g(M)$ , the solution  $\phi$  of  $(*)_t$  satisfies the Harnack type inequality*

$$(0.1) \quad -\frac{1}{\text{Vol}_g(M)} \int_M (\phi - \psi)(\omega_g + \partial\bar{\partial}\phi)^n \leq n \sup_M (\phi - \psi).$$

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Furthermore, there is a constant  $C(t)$  depending only on  $t$  such that for  $t > 0$ , the solution  $\phi$  of  $(*)_t$  satisfies

$$(0.2) \quad -\inf_M(\phi - \psi) \leq n \sup_M(\phi - \psi) + C(t).$$

An interesting and surprising corollary of Theorem 1 is the following.

**Corollary 1.** *There is a universal constant  $C$  such that for any Kähler-Einstein manifold  $(M, g)$  with  $C_1(M) > 0$ , i.e.,  $\text{Ric}(g) = \omega_g$ , and for any  $C^2$  function  $\psi$  with  $\omega_g + \partial\bar{\partial}\psi \geq 0$ ,  $\int_M e^{-\psi} \omega_g^n = \text{Vol}_g(M)$ , the following inequality holds:*

$$(0.3) \quad \sup_M \psi \leq -n \inf_M \psi + C.$$

Note that the constant  $C$  is computable and the inequality (0.3) is sharp; for example, one can consider the case  $(M, g) = (CP^n, \text{Fubini-Study metric})$  to see the sharpness of (0.3).

### 1. The proof of Theorem 1

First, we assume that  $\psi \in C^\infty(M, R)$ ,  $\omega_g + \partial\bar{\partial}\psi > 0$ . Then we can define a new Kähler metric  $\tilde{g}$  such that the associated Kähler form is  $\omega_{\tilde{g}} + \partial\bar{\partial}\psi$ . Put  $\tilde{f} = f + \log(\omega_g^n / \omega_{\tilde{g}}^n) - t\psi$ . Then

$$(1.1) \quad \int_M e^{\tilde{f}} \omega_{\tilde{g}}^n = \int_M e^{f-t\psi} \omega_g^n = \text{Vol}_g(M).$$

Rewrite  $(*)_t$  in terms of  $\omega_{\tilde{g}}$  and  $\tilde{f}$  as follows:

$$(1.2) \quad \begin{aligned} (\omega_{\tilde{g}} + \partial\bar{\partial}(\phi - \psi))^n &= e^{\tilde{f}-t(\phi-\psi)} \omega_{\tilde{g}}^n, \\ (\omega_{\tilde{g}} + \partial\bar{\partial}(\phi - \psi)) &> 0, \text{ on } M. \end{aligned}$$

In [8], in order to show the uniqueness of Kähler-Einstein metrics on a compact Kähler manifold with positive first Chern class, Bando and Mabuchi first prove the solvability of  $(*)_{t'}$  for  $t' \leq t \leq 1$  under the assumption that  $(*)_t$  has a solution. We will apply this idea to the following equations in the first part of our proof. Precisely, we will first prove that the following equations are always solvable for  $0 \leq s \leq t$ :

$$(1.3)_s \quad \begin{aligned} (\omega_{\tilde{g}} + \partial\bar{\partial}\theta)^n &= e^{\tilde{f}-s\theta} \omega_{\tilde{g}}^n, \\ (\omega_{\tilde{g}} + \partial\bar{\partial}\theta) &> 0, \text{ on } M. \end{aligned}$$

As usual, we use the continuity method. Define  $S = \{s \in [0, t] | (1.3)_s \text{ is solvable for } s' \in [s, t]\}$ . Since (1.2) has a solution  $\phi$ ,  $t \in S$  and  $S$  is nonempty. It is sufficient to show that  $S$  is both open and closed. For the openness,

we should estimate the first eigenvalue of the metric  $g_\theta$  associated with the Kähler form  $\omega_g + \partial\bar{\partial}\theta$  for the solution  $\theta$  of (1.3)<sub>s</sub>.

**Lemma 1.1.** *The first nonzero eigenvalue  $\lambda_1(g_\theta)$  is greater than  $s$ .*

*Proof.* By the well-known Bochner identity (see [1]), it suffices to show that  $\text{Ric}(g_\theta)$  is strictly bounded from below by  $s$ . From (1.3)<sub>s</sub>, we have

$$\begin{aligned} \text{Ric}(g_\theta) &= \text{Ric}(\tilde{g}) - \partial\bar{\partial}\tilde{f} + s\partial\bar{\partial}\theta \\ &= \text{Ric}(\tilde{g}) - \partial\bar{\partial}f - \partial\bar{\partial}\log\left(\frac{\omega_g^n}{\omega_{\tilde{g}}^n}\right) + t\partial\bar{\partial}\psi + s\partial\bar{\partial}\theta \\ &= \text{Ric}(g) - \partial\bar{\partial}f + t\partial\bar{\partial}\psi + s\partial\bar{\partial}\theta \\ &= (1-t)\omega_g + (t-s)\omega_{\tilde{g}} + s(\omega_{\tilde{g}} + \partial\bar{\partial}\theta) > s\omega_{g_\theta}. \end{aligned}$$

The first variation of (1.3)<sub>s</sub> at  $\theta$  is  $\Delta_s u = -su$ , where  $\Delta_s$  is the Laplacian of the metric  $g_\theta$ . Lemma 1.1 implies that the linearized operator  $\Delta_s - s$  of (1.3)<sub>s</sub> is invertible; then the openness follows from Implicit Function Theorem.

For the closedness of  $S$ , by the standard theory of elliptic equations [4] and Yau's estimates of higher derivatives for solutions of complex Monge-Ampère equations of type (1.3)<sub>s</sub>, it suffices to estimate  $C^0$ -norms of the solutions of (1.3)<sub>s</sub>.

Suppose that (1.3)<sub>s</sub> is solvable for  $s \in (s_0, t]$  and  $\theta_s$  is the solution. From the proof of the openness of  $S$ , one can actually conclude that  $\{\theta_s\}_{s \in (s_0, t]}$  is a smooth family in  $C^\infty(M, R)$ , i.e.,  $\theta_s$  varies smoothly with  $s$ .

Define, as in [1],

$$I(\theta_s) = \frac{1}{\text{Vol}_g(M)} \int_M \theta_s (\omega_{\tilde{g}}^n - (\omega_{\tilde{g}} + \partial\bar{\partial}\theta_s)^n), \quad J(\theta_s) = \int_0^1 \frac{I(x\theta_s)}{x} dx.$$

**Lemma 1.2** ([1], [2], [7]). (i)  $(n+1)J(\theta_s)/n \leq I(\theta_s) \leq (n+1)J(\theta_s)$ ,

(ii)  $d(I(\theta_s) - J(\theta_s))/ds = -(\text{Vol}_g(M))^{-1} \int_M \theta_s (\Delta_s \dot{\theta}_s) \omega_g^n$ ,

where  $\dot{\theta}_s = d\theta_{s'}/ds'|_{s'=s}$ ,  $g_s$  is the Kähler metric associated with  $\omega_{\tilde{g}} + \partial\bar{\partial}\theta_s$  and  $\Delta_s$  is its Laplacian.

As a corollary, we have the following lemma which was observed by Bando and Mabuchi [2] and the author [7].

**Lemma 1.3.**  *$I(\theta_s) - J(\theta_s)$  is monotonically increasing.*

*Proof.* Differentiate (1.3)<sub>s</sub> with respect to  $s$ :

$$(1.4)_s \quad \Delta_s \dot{\theta}_s = -s\dot{\theta}_s - \theta_s.$$

Substituting (1.4)<sub>s</sub> into the right-hand side of the formula in Lemma 1.2(ii), we obtain

$$(1.5) \quad \frac{d}{ds}(I(\theta_s) - J(\theta_s)) = \frac{1}{\text{Vol}_g(M)} \int_M (\Delta_s \dot{\theta}_s + s\dot{\theta}_s) \Delta_s \dot{\theta}_s \omega_g^n.$$

Write  $\dot{\theta}_s$  in the eigenfunction expansion, i.e.,

$$(1.6) \quad \dot{\theta}_s = \sum_{i=0}^{\infty} a_i u_i,$$

where  $-\Delta_s u_i = \lambda_i u_i$ ,  $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \dots$ .

By Lemma 1.1.,  $\lambda_1 > s$ . Hence,

$$\begin{aligned} \frac{d}{ds}(I(\theta_s) - J(\theta_s)) &= \frac{1}{\text{Vol}_g(M)} \int_M \left( \sum_{i=0}^{\infty} a_i (\lambda_i - s) u_i \right) \left( \sum_{j=0}^{\infty} a_j \lambda_j u_j \right) \omega_{g_s}^n \\ &= \frac{1}{\text{Vol}_g(M)} \sum_{i=0}^{\infty} |a_i|^2 \int_M (\lambda_i - s) \lambda_i |u_i|^2 \omega_{g_s}^n \geq 0, \end{aligned}$$

and the lemma is proved.

In the following, we always denote by  $C$  the constant independent of  $s$ .

**Lemma 1.4.** *There is a constant  $C > 0$  such that for any solution  $\theta_s$  of (1.3)<sub>s</sub>,  $0 < s \leq t$ , we have  $\sup_M |\theta_s| \leq C$ .*

*Proof.* In [7], we define a holomorphic invariant  $\alpha(M)$  on the compact Kähler manifold  $M$  with  $C_1(M) > 0$ . For any  $\lambda < \alpha(M)$ , there is a constant  $C_\lambda$ , which may depend on the metric  $\tilde{g}$ , such that

$$(1.7) \quad \int_M e^{-\lambda(u - \sup_M u)} dV_{\tilde{g}} \leq C_\lambda \quad \text{for } u \in C^2(M, \mathbb{R}), \omega_{\tilde{g}} + \partial\bar{\partial}u \geq 0.$$

In case  $s \in (0, \alpha(M)/(n+2)]$ ,  $\int_M e^{-(n+1)s(\theta_s - \sup_M \theta_s)} dV_{\tilde{g}} \leq C$  for a constant  $C$ . For  $p > 0$ ,

$$\begin{aligned} &\int_M e^{-p(\theta_s - \sup_M \theta_s)} (e^{\tilde{f} - s\theta_s} - 1) dV_{\tilde{g}} \\ &= \int_M e^{-p(\theta_s - \sup_M \theta_s)} (\omega_{g_s}^n - \omega_{\tilde{g}}^n) \\ &= \int_M e^{-p(\theta_s - \sup_M \theta_s)} \partial\bar{\partial}(\theta_s - \sup_M \theta_s) (\omega_{g_s}^{n-1} + \omega_{g_s}^{n-2} \wedge \omega_{\tilde{g}} + \dots + \omega_{\tilde{g}}^{n-1}) \\ &= \frac{4}{p} \int_M \partial(e^{-p(\theta_s - \sup_M \theta_s)/2}) \wedge \bar{\partial}(e^{-p(\theta_s - \sup_M \theta_s)/2}) \\ &\quad \wedge (\omega_{g_s}^{n-1} + \omega_{g_s}^{n-2} \wedge \omega_{\tilde{g}} + \dots + \omega_{\tilde{g}}^{n-1}) \\ &\geq \frac{4}{p} \int_M |\tilde{\nabla}(e^{-p(\theta_s - \sup_M \theta_s)/2})|^2 dV_{\tilde{g}} \\ &\geq \frac{4c}{p} \left( \int_M e^{-np(\theta_s - \sup_M \theta_s)/(n-1)} dV_{\tilde{g}} \right)^{(n-1)/n} \\ &\quad - \frac{4C}{p} \int_M e^{-p(\theta_s - \sup_M \theta_s)} dV_{\tilde{g}}, \end{aligned}$$

where  $c$  is the Sobolev constant, depending only on  $(M, \tilde{g})$ . Using Hölder inequality on the left-handed side of the above, we have

$$(1.8) \quad \left( \int_M e^{-np(\theta_s - \sup_M \theta_s)/(n-1)} dV_{\tilde{g}} \right)^{(n-1)/n} \leq Cp \left( \int_M e^{-(n+1)p(\theta_s - \sup_M \theta_s)/n} dV_{\tilde{g}} \right)^{n/(n+1)} \cdot \left( \left( \int_M e^{-(n+1)s\theta_s} dV_{\tilde{g}} \right)^{1/n+1} + 1 \right).$$

Now  $\sup_M \theta_s \geq 0$ , since

$$\text{Vol}_{\tilde{g}}(M) = \int_M \omega_{\tilde{g}}^n = \int_M \omega_{g_s}^n = \int_M e^{\tilde{f} - s\theta_s} dV_{\tilde{g}} \geq e^{-s \sup_M \theta_s} \int_M e^{\tilde{f}} dV_{\tilde{g}} = e^{-s \sup_M \theta_s} \text{Vol}_{\tilde{g}}(M).$$

Then

$$(1.9) \quad \left( \int_M e^{-(n+1)\theta_s} dV_{\tilde{g}} \right)^{n+1} \leq \left( \int_M e^{-(n+1)s(\theta_s - \sup_M \theta_s)} dV_{\tilde{g}} \right)^{n+1} \leq C^{n+1}.$$

Substituting (1.9) into (1.8), we have

$$(1.10) \quad \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{np/(n-1)} \leq C^{1/p} p^{1/p} \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{(n+1)p/n}.$$

Put  $p_0 = ns$  and  $p_{m+1} = p_m n^2 / (n^2 - 1)$ . Then

$$\begin{aligned} \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{p_{m+1}} &\leq C^{(n/(n-1))(1/p_{m+1})} \left( \frac{n-1}{n} p_{m+1} \right)^{(n/(n-1))/(1/p_{m+1})} \\ &\quad \cdot \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{p_m} \\ &\leq (C p_m)^{1/p_m} \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{p_m} \\ &\quad \dots \\ &\leq C^{1/ns} \sum_{m=0}^{\infty} \left( \frac{n^2 - 1}{n^2} \right)^m \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{p_0} \\ &\quad \cdot \exp \left( \frac{1}{np} \sum_{m=1}^{\infty} \left( \frac{n^2 - 1}{n^2} \right)^m \left( m \log \frac{n^2}{n^2 - 1} + \log(ns) \right) \right) \\ &\leq C, \end{aligned}$$

and it follows that

$$-\inf_M (\theta_s - \sup_M \theta_s) = \log \left( \lim_{m \rightarrow \infty} \left| e^{-(\theta_s - \sup_M \theta_s)} \right|_{p_m} \right) \leq C,$$

i.e., for  $s \in (0, \alpha(M)/(n+2)]$ ,  $\sup_M |\theta_s| \leq C$ .

In the case  $s \geq \alpha(M)/(n+2)$ ,  $\text{Ric}(g_s) \geq s \geq \alpha(M)/(n+2)$ . Then by Bochner's identity and results of Croke [3] and P. Li [5], we have both the Sobolev inequality and the Poincaré inequality with their constants uniformly bounded on  $(M, g_s)$ . Since  $-\Delta_s \theta_s \geq -n$ , the standard Moser iteration implies that

$$(1.11) \quad -\inf_M \theta_s \leq C \int_M (-\theta_s) \omega_{g_s}^n + C,$$

(also see [7] for details).

On the other hand, by Green's formula on  $(M, \tilde{g})$ , it follows that

$$(1.12) \quad \sup_M \theta_s \leq \int_M \theta_s \omega_{\tilde{g}}^n + C.$$

By Lemmas 1.3 and 1.2(i),

$$(1.13) \quad I(\theta_s) \leq (n+1)(I(\theta_s) - J(\theta_s)) \leq (n+1)(I(\theta_t) - J(\theta_t)) \leq C.$$

Since  $\int_{\{\theta_s > 0\}} \theta_s e^{\tilde{f} - s\theta_s} dV_{\tilde{g}}$  and  $\int_{\{\theta_s < 0\}} (-\theta_s) dV_{\tilde{g}}$  are obviously bounded by a constant  $C$  independent of  $s$ , the Lemma follows from (1.11)–(1.13) and the definition of  $I(\theta_s)$ .

Now the closeness of  $S$  follows from the above lemma. Hence,  $(1.3)_s$  is solvable for  $0 \leq s \leq t$ . Then there is a smooth family of  $\{\theta_s\}_{0 \leq s}$  such that  $\theta_t = \phi - \psi$ . By Lemma 1.2(ii) and (1.4)<sub>s</sub>, we have

$$\begin{aligned} \frac{d}{ds}(I(\theta_s) - J(\theta_s)) &= -\frac{1}{\text{Vol}_{\tilde{g}}(M)} \int_M \theta_s (-s\dot{\theta}_s - \theta_s) \omega_{g_s}^n, \\ &= -\frac{1}{\text{Vol}_{\tilde{g}}(M)} \frac{d}{ds} \left( \int_M \theta_s \omega_{g_s}^n \right) + \frac{1}{\text{Vol}_{\tilde{g}}(M)} \int_M \dot{\theta}_s \omega_{g_s}^n. \end{aligned}$$

Differentiating  $\text{Vol}_{\tilde{g}}(M) = \int_M e^{\tilde{f} - s\theta_s} dV_{\tilde{g}}$  gives

$$(1.14) \quad \int_M (-s\dot{\theta}_s - \theta_s) e^{\tilde{f} - s\theta_s} dV_{\tilde{g}} = 0.$$

Hence,

$$\frac{d}{ds}(I(\theta_s) - J(\theta_s)) = \frac{1}{s \text{Vol}_{\tilde{g}}(M)} \frac{d}{ds} \left( s \int_M (-\theta_s) e^{\tilde{f} - s\theta_s} dV_{\tilde{g}} \right);$$

i.e.,

$$(1.15) \quad \begin{aligned} \frac{d}{ds}(s(I(\theta_s) - J(\theta_s))) - (I(\theta_s) - J(\theta_s)) \\ = \frac{d}{ds} \left( \frac{s}{\text{Vol}_{\tilde{g}}(M)} \int_M -\theta_s e^{\tilde{f} - s\theta_s} dV_{\tilde{g}} \right). \end{aligned}$$

Note that

$$\begin{aligned}
 I(\theta_s) - J(\theta_s) &\geq \frac{1}{n+1} I(\theta_s) = \frac{1}{(n+1) \text{Vol}_{\bar{g}}(M)} \int_M \theta_s (\omega_{\bar{g}}^n - \omega_{g_s}^n) \\
 &= \frac{1}{(n+1) \text{Vol}_{\bar{g}}(M)} \int_M \partial\theta_s \wedge \bar{\partial}\theta_s \\
 &\quad \wedge (\omega_{\bar{g}}^{n-1} + \omega_{\bar{g}}^{n-2} \wedge \omega_{g_s} + \dots + \omega_{g_s}^{n-1}) \geq 0.
 \end{aligned}$$

Then it follows from (1.15) and Lemma 1.2(i) that

$$\begin{aligned}
 (1.16) \quad \frac{1}{\text{Vol}_{\bar{g}}(M)} \int_M (-\theta_t) \omega_{g_s}^n \leq I(\theta_t) - J(\theta_t) &\leq \frac{n}{n+1} I(\theta_t) \\
 &= \frac{n}{n+1} \frac{1}{\text{Vol}_{\bar{g}}(M)} \int_M \theta_t (\omega_{\bar{g}}^n - \omega_{g_t}^n),
 \end{aligned}$$

i.e.,

$$\begin{aligned}
 -\frac{1}{\text{Vol}_{\bar{g}}(M)} \int_M (\phi - \psi) (\omega_g + \partial\bar{\partial}\phi)^n &\leq \frac{n}{\text{Vol}_{\bar{g}}(M)} \int_M (\phi - \psi) \omega_{\bar{g}}^n \\
 &\leq n \sup_M (\phi - \psi),
 \end{aligned}$$

which is just (0.1). The inequality (0.2) follows from Moser’s iteration and the fact that  $\text{Ric}(g_t) \geq t > 0$ . Hence Theorem 1 is proved. (We refer the reader to the proof of Lemma 1.4 for details.)

### 2. The proof of Corollary 1

From (0.2) in Theorem 1, for any  $\psi \in C^2(M, R)$  with  $\omega_g + \partial\bar{\partial}\psi \geq 0$  and  $\int_M e^{-\psi} \omega_g^n = \text{Vol}_g(M)$ , we have

$$(2.1) \quad -\inf_M (\phi - \psi) \leq n \sup_M (\phi - \psi) + C(1),$$

where  $\phi$  is the solution of  $(*)_1$  and  $C(1)$  is a universal constant. Note that here  $f \equiv 0$ , since  $g$  has been a Kähler-Einstein metric. This implies that  $\phi \equiv 0$  is a solution of  $(*)_1$ . For  $\phi = 0$ , (2.1) becomes

$$-\inf_M (-\psi) \leq n \sup_M (-\psi) + C(1).$$

Because  $\inf_M (-\psi) = -\sup_M (\psi)$  and  $\sup_M (-\psi) = -\inf_M \psi$ , Corollary 1 is proved.

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